Geometrical structure of entangled states and secant variety

Hoshang Heydari
Institute of Quantum Science, Nihon University,
1-8 Kanda-Surugadai, Chiyoda-ku, Tokyo 101-8308, Japan
Email: hoshang@edu.cst.nihon-u.ac.jp

Abstract

We show that the secant variety of the Segre variety gives useful information about the geometrical structure of an arbitrary multipartite quantum system. In particular, we investigate the relation between arbitrary bipartite and three-partite entangled states and this secant variety. We also discuss the geometry of an arbitrary general multipartite state.

1 Introduction

Recently, the geometry and topology of entanglement has got more attention and we know more about the geometrical structure of pure multipartite entangled quantum states. We have also managed to construct some useful measures of entanglement based on these underlying geometrical structures. However, we know less about the geometrical structure of an arbitrary multipartite quantum state and there is a need for further investigation on these states. Concurrence is a measure of entanglement which is directly related to the entanglement of formation [1]. Its geometrical structure is hidden in a map called Segre embedding [2, 3, 4, 5]. The Segre variety is generated by the quadratic polynomials that correspond to the separable set of pure multipartite states. We can construct a measure of entanglement for bipartite and three-partite states based on the Segre variety [4]. We can also construct a measure of entanglement for general pure multipartite states based on a modification of the Segre variety by adding similar quadratic polynomials [5]. In this paper, we will establish a relation between the secant variety of the Segre variety and multipartite states. For example we show that the concurrence of arbitrary bipartite and three-partite entangled states are equivalent to the secant variety of the Segre variety. We also generalize our result for a measure of entanglement for an arbitrary multipartite state. In section 2, we will define the complex projective variety. We also introduce the Segre embedding and the Segre variety for general pure multipartite states. In following section 3, we will define and discuss the secant variety of a projective variety. In section 4, we investigate the secant variety of the Segre variety, which is of central importance in this paper. Moreover, we investigate relation and relevance of the secant variety of the Segre variety as geometrical structure of entangled and separable states. For example, we show that this variety defines the space of the concurrence of a mixed state. Finally, in section 5, we expand our result to an arbitrary multipartite

state. As usual, we denote a general, composite quantum system with m subsystems as $\mathcal{Q} = \mathcal{Q}_m(N_1, N_2, \ldots, N_m) = \mathcal{Q}_1\mathcal{Q}_2\cdots\mathcal{Q}_m$, consisting of the pure states $|\Psi\rangle = \sum_{k_1=1}^{N_1}\sum_{k_2=1}^{N_2}\cdots\sum_{k_m=1}^{N_m}\alpha_{k_1,k_2,\ldots,k_m}|k_1, k_2, \ldots, k_m\rangle$ and corresponding to the Hilbert space $\mathcal{H}_{\mathcal{Q}} = \mathcal{H}_{\mathcal{Q}_1}\otimes\mathcal{H}_{\mathcal{Q}_2}\otimes\cdots\otimes\mathcal{H}_{\mathcal{Q}_m}$, where the dimension of the jth Hilbert space is $N_j = \dim(\mathcal{H}_{\mathcal{Q}_j})$. We are going to use this notation throughout this paper. In particular, we denote a pure two-qubit state by $\mathcal{Q}_2^p(2,2)$. Concurrence is a widely used measure of entanglement which have been successfully applied to many different field of finite quantum systems. It also gives an analytical expression for entanglement of formation for bipartite quantum systems. In Wootter's definition of concurrence for a two-qubit state [1], the tilde operation is an example of conjugation, i.e., an antiunitary operator. Based on this observation, Uhlmann [6] generalized the concept of concurrence. Uhlmann considered an arbitrary conjugation acting on an arbitrary Hilbert space. For example concurrence for a pure quantum system $\mathcal{Q}_2^p(N_1, N_2)$ is defined by

$$\mathcal{C}\left(\mathcal{Q}_2^p(N_1, N_2)\right) = \sqrt{|\langle\Psi|\Theta\Psi\rangle|}, \tag{1.0.1}$$

where Θ is an antilinear operator and satisfies $\Theta^2 = I$. Moreover, the concurrence of a quantum system $\mathcal{Q}_2(N_1, N_2)$ is defined by

$$\mathcal{C}\left(\mathcal{Q}_2(N_1, N_2)\right) = \inf\sum_i p_i\mathcal{C}^i\left(\mathcal{Q}_2^p(N_1, N_2)\right), \tag{1.0.2}$$

where the infimum is taken over all pure state decompositions. If we have a quantum system $\mathcal{Q}_2(2,2)$, where Θ is a tilde operation, then concurrence coincides with Wootter's formula for concurrence of two-qubit states.

2 Complex projective variety

In this section, we review some basic definition of complex projective variety. The general reference on projective algebraic geometry can be found in [7, 8, 9]. Let $C[z] = C[z_1, z_2, \ldots, z_n]$ denote the polynomial algebra in n variables with complex coefficients. Then, given a set of q polynomials $\{h_1, h_2, \ldots, h_q\}$ with $h_i \in C[z]$, we define a complex affine variety as

$$\mathcal{V}_{\mathbb{C}}(h_1, h_2, \ldots, h_q) = \{P \in \mathbb{C}^n : h_i(P) = 0\ \forall\ 1 \leq i \leq q\}, \tag{2.0.3}$$

where $P = (a_1, a_2, \ldots, a_n)$ is called a point of $\mathbb{C}^n$ and the a_i are called the coordinates of P. A complex projective space $\mathbb{P}^n$ is defined to be the set of lines through the origin in $\mathbb{C}^{n+1}$, that is, $\mathbb{P}^n = \mathbb{C}^{n+1} - 0/\sim$, where the equivalence relation $\sim$ is defined as follow; $(x_1, \ldots, x_{n+1}) \sim (y_1, \ldots, y_{n+1})$ for $\lambda \in \mathbb{C} - 0$, where $y_i = \lambda x_i$ for all $0 \leq i \leq n+1$. Given a set of homogeneous polynomials $\{h_1, h_2, \ldots, h_q\}$ with $h_i \in C[z]$, we define a complex projective variety as

$$\mathcal{V}(h_1, \ldots, h_q) = \{O \in \mathbb{P}^n : h_i(O) = 0\ \forall\ 1 \leq i \leq q\}, \tag{2.0.4}$$

where $O = [a_1, a_2, \ldots, a_{n+1}]$ denotes the equivalent class of points $\{\alpha_1, \alpha_2, \ldots, \alpha_{n+1}\} \in \mathbb{C}^{n+1}$. We can view the affine complex variety $\mathcal{V}_{\mathbb{C}}(h_1, h_2, \ldots, h_q) \subset \mathbb{C}^{n+1}$ as a complex cone over the complex projective variety $\mathcal{V}(h_1, h_2, \ldots, h_q)$.

As an important example of projective variety we will discuss the Segre variety. For a multipartite quantum system $\mathcal{Q}(N_1, \ldots, N_m)$, let $\overline{N} = (N_1 -$

$1, \ldots, N_m - 1)$ and $V_1, V_2, \ldots, V_m$ be vector spaces over the field of complex numbers $\mathbb{C}$, where $\dim V_j = N_j$. That is, we have $\mathbb{P}^{N_j - 1} = \mathbb{P}(V_j)$ for all j. Then we define a Segre map by

$$\mathcal{S}_{N_1,N_2,\ldots,N_m} : \mathbb{P}^{N_1-1} \times \mathbb{P}^{N_2-1} \times \cdots \times \mathbb{P}^{N_m-1} \longrightarrow \mathbb{P}^{\mathcal{N}-1}, \tag{2.0.5}$$

where $\mathcal{N} = \prod_{j=1}^{m} N_j$. This map is based on the canonical multilinear map

$$\begin{array}{ccc} V_1 \times V_2 \times \cdots \times V_m & \rightarrow & V_1 \otimes V_2 \otimes \cdots \otimes V_m \\ v_1 \times v_2 \times \cdots \times v_m & \mapsto & v_1 \otimes v_2 \otimes \cdots \otimes v_m \end{array} \tag{2.0.6}$$

Thus, we have $\mathbb{P}^{\mathcal{N}-1} = \mathbb{P}(V_1, \ldots, V_m)$. The Segre variety $\mathfrak{S}_{\overline{N}} = \mathrm{Im}(\mathcal{S}_{N_1,N_2,\ldots,N_m})$ is defined to be the image of the Segre embedding. By definition, the Segre variety is formed by the set of all classes of decomposable tensors in $\mathbb{P}^{\mathcal{N}-1}$. For a quantum system $\mathcal{Q}(N_1, \ldots, N_m)$, the Segre variety is given by

$$\begin{aligned} \mathfrak{S}_{\overline{N}} \quad = \quad & \bigcap_{\forall j} \mathcal{V}(\alpha_{k_1,k_2,\ldots,k_m}\alpha_{l_1,l_2,\ldots,l_m} \\ & -\alpha_{k_1,k_2,\ldots,k_{j-1},l_j,k_{j+1},\ldots,k_m}\alpha_{l_1,l_2,\ldots,l_{j-1},k_j,l_{j+1},\ldots,l_m}). \end{aligned} \tag{2.0.7}$$

We can also partition the Segre embedding as follows:

$$\begin{tikzcd} (\mathbb{P}^{N_1-1} \times \cdots \times \mathbb{P}^{N_l-1}) \times (\mathbb{P}^{N_l-1} \times \cdots \times \mathbb{P}^{N_m-1}) \arrow[r] \arrow[d, "\mathcal{S}_{N_1,N_2,\ldots,N_m}"'] & \mathbb{P}^{\mathcal{M}_1} \times \mathbb{P}^{\mathcal{M}_2} \arrow[d, "I \otimes I"'] \\ \mathbb{P}^{N_1 N_2 \cdots N_m - 1} & \mathbb{P}^{\mathcal{M}_1} \times \mathbb{P}^{\mathcal{M}_2} \arrow[l, "\mathcal{S}_{\mathcal{M}_1,\mathcal{M}_2}"'] \end{tikzcd}$$

where $\mathcal{M}_1 = N_1 N_2 \ldots N_l - 1$,$\mathcal{M}_2 = N_{l+1}N_{l+2}\ldots N_m - 1$ and $(\mathcal{M}_1 + 1)(\mathcal{M}_2 + 1) = N_1 N_2 \ldots N_m$. For the Segre variety, which is represented by a completely decomposable tensors, the above diagram commutate. Let $\mathbb{X} \subset \mathbb{P}^N$. Then there are two important subsets of $\mathbb{P}^N$; the secant variety $\mathfrak{Sec}(\mathbb{X})$, which is defined to be the closure of the set of point lying on secant $\overline{x_1 x_2}$, where x_1 and x_2 are distinct points of $\mathbb{X}$. The second one $\mathfrak{Tan}(\mathbb{X})$ is the union of the projective tangent spaces. In the next section we will discuss the secant variety of a projective variety.

3 Secant variety

The secant variety of a projective variety has been studied in algebraic geometry and some recent references include [10, 11]. The k-th secant variety $\mathfrak{Sec}_k(\mathbb{X})$ of $\mathbb{X} \subset \mathbb{P}^M$ with $\dim \mathbb{X} = d$ is defined to be the closure of the union of k-dimensional linear subspaces of $\mathbb{P}^M$ determined by general $k+1$ points on $\mathbb{X}$

$$\mathfrak{Sec}_k(\mathbb{X}) = \overline{\bigcup \{\text{all secant } \mathbb{P}^k \text{ 's to } \mathbb{X}\}}, \tag{3.0.8}$$

where for $P_0, P_1 \ldots P_k \in \mathbb{X}$, we have $\mathbb{P}^k = \langle P_0, P_1 \ldots P_k \rangle$. Moreover, the dimension of $\mathfrak{Sec}_k(\mathbb{X})$ satisfies

$$\dim \mathfrak{Sec}_k(\mathbb{X}) \leq \min\{M^{'}, (k+1)(d+1) - 1\}, \tag{3.0.9}$$

where $M^{'}$ is the dimension of the linear subspace spanned by $\mathbb{X}$. The subvariety $\mathbb{X}$ is called k-defect when $\dim \mathfrak{Sec}_k(\mathbb{X}) < \min\{M^{'}, (k+1)(d+1)-1\}$. For example, the secant variety of Segre variety $\mathfrak{Sec}_k(\mathfrak{S}_{\overline{N}})$ is the closure of the set of classes of those tensor products which can be written as the sum of at most $k+1$ decomposable tensor products. Thus, the secant variety of Segre variety $\mathfrak{Sec}_k(\mathfrak{S}_{\overline{N}})$ gives some useful information about the geometry of entangled and separable mixed multipartite states.

4 Secant variety of the Segre variety and concurrence

In this section, we investigate the secant variety of the Segre variety and show that the geometry of concurrence for mixed bipartite and three-partite states is given by this variety. In the next section, we will discuss the secant variety of variety for an arbitrary multipartite state. For bipartite quantum system $\mathcal{Q}(N_1, N_2)$, the Segre variety $\mathfrak{S}_{\overline{N}}$ is the variety of $N_1 \times N_2$ matrices of rank 1. Thus the secant variety $\mathfrak{Sec}_k(\mathfrak{S}_{N_1,N_2})$ is the matrices of rank less than k and $k = N_1$ is the least integer for which $\mathfrak{Sec}_k(\mathfrak{S}_{N_1,N_2}) = \mathbb{P}^{N_1 N_2 - 1}$. The Segre variety has two rulings by the families of linear spaces $v \otimes \mathbb{P}(W)$ and $\mathbb{P}(V) \otimes w$ for all $v \in V$ and $w \in W$. The Segre variety can be seen as decomposable tensors in $\mathbb{P}(V) \otimes \mathbb{P}(W)$. The k-fold secant plane to the Segre variety is given by the tensor of rank k. For example, a tensor which can be written as $\sum_{i=1}^{k} v_i \otimes w_i = v_1 \otimes w_1 + v_2 \otimes w_2 + \cdots + v_k \otimes w_k$. As an example, we will discuss the Secant variety of the Segre variety $\mathfrak{Sec}_k(\mathfrak{S}_{(3,3)})$ of quantum system $\mathcal{Q}_2(3,3)$. For this quantum system, the Segre variety is given by $\mathfrak{S}_{(3,3)} = \bigcap_{k_1,l_1,k_2,l_2=1}^{3} \mathcal{V}(\alpha_{k_1,k_2}\alpha_{l_1,l_2} - \alpha_{k_1,l_2}\alpha_{l_1,k_2})$. Moreover, we have $\dim \mathfrak{Sec}_1(\mathfrak{S}_{(3,3)}) = (3+3)(1+1) - (1+1)^2 - 1 = 7$, but the expected dimension was $\dim \mathfrak{Sec}_1(\mathfrak{S}_{(3,3)}) \leq \min\{M^{'}, (k+1)(d+1)-1\} = \min\{8, (1+1)(4+1)-1\} = 8$. We have expected that the secant variety $\mathfrak{Sec}_1(\mathfrak{S}_{(3,3)})$ does fill the enveloping space and this is an example of a deficient Segre variety. For bipartite systems, if we assume that $N_1 < N_2$, then for all $1 \leq k < N_1$ the secant variety $\mathfrak{Sec}_k(\mathfrak{S}_{(N_1,N_2)})$ has dimension less than the expected dimension and the least k for which $\mathfrak{Sec}_k(\mathfrak{S}_{(N_1,N_2)})$ fills its enveloping space is $k = N_1$. Next, we write the concurrence of a quantum system $\mathcal{Q}_2(N_1, N_2)$ as follows

$$
\begin{aligned}
\mathcal{C}\left(\mathcal{Q}_2(N_1,N_2)\right) &= \inf \sum_i p_i \mathcal{C}\left(\Psi_i\right) \qquad (4.0.10) \\
&= \inf \sum_i p_i \left(\mathcal{N} \sum_{k_1,l_1=1}^{N_1} \sum_{k_2,l_2=1}^{N_2} \left| \alpha^i_{k_1,k_2}\alpha^i_{l_1,l_2} - \alpha^i_{k_1,l_2}\alpha^i_{l_1,k_2} \right|^2 \right)^{\frac{1}{2}} \\
&\simeq \inf \left(\sim \text{sum of all decom. ten. in } \mathbb{P}^{N_1N_2-1} \right) \\
&\simeq \inf \mathfrak{Sec}_k(\mathfrak{S}_{N_1,N_2}),
\end{aligned}
$$

where N is a normalization constant. From this expression we can see that the geometry of concurrence of arbitrary bipartite state is given by the secant variety of the Segre variety. Moreover, the geometry of arbitrary three-partite states can be given by this secant variety, since we can construct a measure for

three-partite states based on the Segre variety in the same way as we did for bipartite states. The generalized concurrence for such a state is given by

$$\begin{aligned}\mathcal{C}(\mathcal{Q}^p_3(N_1,N_2,N_3)) &= (\mathcal{N}\sum^{m=3}_{k_1,l_1;k_2,l_2;k_3,l_3}\sum_{\forall j}|\alpha_{k_1,k_2,\ldots,k_m}\alpha_{l_1,l_2,\ldots,l_m} \\ &\quad -\alpha_{k_1,k_2,\ldots,k_{j-1},l_j,k_{j+1},\ldots,k_m}\alpha_{l_1,l_2,\ldots,l_{j-1},k_j,l_{j+1},\ldots,l_m}|^2)^{\frac{1}{2}}.\end{aligned} \tag{4.0.11}$$

From this equation and the discussion about the concurrence of bipartite states, we have

$$\begin{aligned}\mathcal{C}(\mathcal{Q}_3(N_1,N_2,N_3)) &= \inf\sum_i p_i\mathcal{C}^i(\mathcal{Q}^p_3(N_1,N_2,N_3)) \\ &\simeq \inf \mathfrak{Sec}_k(\mathfrak{S}_{N_1,N_2,N_3}).\end{aligned}$$

We can also connect the secant variety of the Segre variety to the separable set of multipartite states $\mathcal{Q}(N_1,N_2,\ldots,N_m)$ based on the relation between perfect codes and Secant variety of the Segre variety. The existence of perfect codes can be proved based on finite fields with q elements. The perfect code exist only for the following parameters: q is a prime power, $t=\frac{q^l-1}{q-1}$ for $l\geq 2$ and $k=q^{t-l}$. Let us look at some examples of this kind. Let $q=2$, $t=2^l-1$, and $k=q^{t-l}$, where l is a positive number. Then for the Segre embedding

$$\mathcal{S}_{2,2,\ldots,2}:\overbrace{\mathbb{P}^1\times\mathbb{P}^1\times\cdots\times\mathbb{P}^1}^{t-\text{times}}\longrightarrow\mathbb{P}^{2^t-1}, \tag{4.0.12}$$

the secant variety of the corresponding Segre variety $\mathfrak{Sec}_{k-1}(\mathfrak{S}_t)=\mathbb{P}^{2^t-1}$ which fits exactly into its enveloping space. Thus, all $\mathfrak{Sec}_{k-1}(\mathfrak{S}_t)$ have the expected dimension. This secant variety coincide with the space of separable mixed multi-qubits states $\mathcal{Q}(2,2,\ldots,2)$.

Next, let q be a prime power. Then for any $l\geq 1$, $t=\frac{q^l-1}{q-1}$, and the Segre embedding

$$\mathcal{S}_{q,q,\ldots,q}:\overbrace{\mathbb{P}^{q-1}\times\mathbb{P}^{q-1}\times\cdots\times\mathbb{P}^{q-1}}^{t-\text{times}}\longrightarrow\mathbb{P}^{q^t-1}, \tag{4.0.13}$$

the secant variety of the Segre variety $\mathfrak{Sec}_{k-1}(\mathfrak{S}_t)=\mathbb{P}^{q^t-1}$ gives information on the geometry of the entangled and separable sets of an arbitrary quantum system $\mathcal{Q}(q,q,\ldots,q)$.

5 Secant variety and arbitrary general multipartite state

Recently, we have proposed a measure of entanglement for general pure multipartite states as [5]

$$\begin{aligned}&\mathcal{F}(\mathcal{Q}^p_m(N_1,\ldots,N_m))=(\mathcal{N}\sum_{\forall\sigma\in\mathrm{Perm}(u)}\sum_{k_j,l_j,j=1,2,\ldots,m} \\ &|\alpha_{k_1k_2\ldots k_m}\alpha_{l_1l_2\ldots l_m}-\alpha_{\sigma(k_1)\sigma(k_2)\ldots\sigma(k_m)}\alpha_{\sigma(l_1)\sigma(l_2)\ldots\sigma(l_m)}|^2)^{\frac{1}{2}},\end{aligned} \tag{5.0.14}$$

where $\sigma \in \mathrm{Perm}(u)$ denotes all possible sets of permutations of indices for which $k_1k_2\ldots k_m$ are replace by $l_1l_2\ldots l_m$, and u is the number of indices to permute. By construction this measure of entanglement vanishes on product states and it is also invariant under all possible permutations of indices. Note that the first set of permutations defines the Segre variety, but there are also additional complex projective variety embedded in $\mathbf{CP}^{\mathcal{N}-1}$ which are defined by other sets of permutations of indices in equation (5.0.17). We can also apply the same procedure as in the case of the concurrence to define a measure of entanglement for arbitrary multipartite states

$$\begin{aligned}\mathcal{F}(\mathcal{Q}_m(N_1,\ldots,N_m)) &= \inf\sum_i p_i\mathcal{F}^i(\mathcal{Q}_m^p(N_1,\ldots,N_m))\\ &= \inf\sum_i p_i(\mathcal{N}\sum_{\forall\sigma\in\mathrm{Perm}(u)}\sum_{k_j,l_j,j=1,2,\ldots,m}\\ &|\alpha^i_{k_1k_2\ldots k_m}\alpha^i_{l_1l_2\ldots l_m}-\alpha^i_{\sigma(k_1)\sigma(k_2)\ldots\sigma(k_m)}\alpha^i_{\sigma(l_1)\sigma(l_2)\ldots\sigma(l_m)}|^2)^{\frac{1}{2}}.\end{aligned} \tag{5.0.15}$$

Next, for a quantum system $\mathcal{Q}(N_1,\ldots,N_m)$ we define the variety

$$\begin{aligned}\mathfrak{T}_{\overline{N}} = \bigcap_{\forall\sigma\in\mathrm{Perm}(u),k_j,l_j,j=1,2,\ldots,m} &\mathcal{V}(\alpha_{k_1,k_2,\ldots,k_m}\alpha_{l_1,l_2,\ldots,l_m}\\ &-\alpha_{\sigma(k_1)\sigma(k_2)\ldots\sigma(k_m)}\alpha_{\sigma(l_1)\sigma(l_2)\ldots\sigma(l_m)}).\end{aligned} \tag{5.0.16}$$

which include the Segre variety. Now, based on our discussion about the concurrence of bipartite and three-partite states, we conclude that

$$\mathcal{F}(\mathcal{Q}_m(N_1,\ldots,N_m)) \simeq \inf\mathfrak{Sec}_k(\mathfrak{T}_{\overline{N}}).$$

This equivalence relation establish a relationes between the geometrical structure of a measure of entanglement for arbitrary general multipartite states and the secant variety $\mathfrak{Sec}_k(\mathfrak{T}_{\overline{N}})$.

We have established a connection between pure mathematics and fundamental quantum mechanics with some applications in the field of quantum information and computing. We have introduced and discussed the secant variety of the Segre variety. But the secant varieties are still subject of research in algebraic geometry. For example, there are still many fundamental open questions about the secant variety of the Segre variety. However, we hope that this geometrical structure may give us some hint to how to solve the problem of quantifying entanglement of an arbitrary multipartite system.

Acknowledgments: The author gratefully acknowledges the financial support of the Japan Society for the Promotion of Science (JSPS).

References

[1] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).

[2] D. C. Brody and L. P. Hughston, Journal of Geometry and Physics **38**, 19 (2001).

[3] A. Miyake, Phys. Rev. A **67**, 012108 (2003).

[4] H. Heydari and G. Björk, J. Phys. A: Math. Gen. **38** (2005) 3203-3211.

[5] H. Heydari, J. Phys. A: Math. Gen. **39** (2006) 9839-9844.

[6] A. Uhlmann, Phys. Rev. A **62**, 032307 (2000).

[7] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley and Sons, New York, 1978.

[8] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, New York, 1977.

[9] D. Mumford, *Algebraic Geometry I, Complex Projective Varieties*, Springer-Verlag, Berlin, 1976.

[10] M. V. Catalisano , A. V. Geramita, and A. Gimigliano, Journal of linear Algebra and Its Applicatons **355** (2002), 263-285.

[11] M. V. Catalisano , A. V. Geramita, and A. Gimigliano, Proc. Amer. Math. Soc. **133** (2005), no. 3, 633-642.